EFFECT OF STEEPNESS OF PULSE FRONTS ON THE RESPONSE OF DIFFERENTIATING AND INTEGRATING CIRCUITS*

By BIMAL KRISHNA BHATTACHARYYA

INSTITUTE OF NUCLEAR PHYSICS, CALCUTTA UNIVERSITY

(Received for publication, February, 13, 1953)

ABSTRACT. The responses of integrating and differentiating circuits to a pulse front that rises linearly upto a certain time and then flattens out have been studied. Generally, the analysis of such circuits is made by assuming the input pulse to be a step-function. That the responses obtained by such an analysis are much different from those found in the case of a linearly rising pulse that flattens out after a certain time, has been shown clearly in both cases of cascaded differentiating and integrating circuits. When the rise time is very short, the output pulse shape becomes almost similar to that in the case of step-function input pulse. As the rise time and the number of cascaded sections increase, the magnitude as well as the rate of rise of the integrated output pulse sharply begins to diminish. In the case of differentiation, the maximum output voltage depends very much upon the rise time.

INTRODUCTION

In calculating the responses of pulsed circuits it is the normal practice to assume the input voltage to be a unit step-function (figure 1) which is defined to be

$$e(t) = 0 \qquad t<0$$
$$= 1 \qquad t>0 \qquad \ldots\ldots\ldots\ldots (1)$$

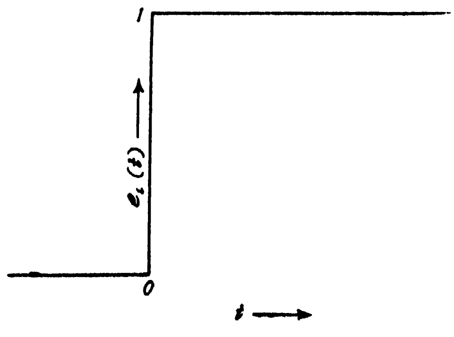

FIG. 1

A step function of unit height.

It is seen from the figure that the voltage changes instantaneously from one value to another. So the wavefront has an infinite slope as indicated by the function.

But it is not physically possible for a voltage or current to jump from one value to another instantaneously. So the responses of pulsed circuits calculated by taking the input voltage to be a step function do not indicate physically accurate pictures. Though the assumption of a perfectly square pulse front is a good approximation of a sharply rising pulse, yet it sometimes leads to erroneous results

* Communicated by Prof. M. N. Saha, F.R.S.

as pointed out by Rubin (1952) and Schwartz (1946) in the case of analysis of pulse differentiation.

A sharply rising pulse can be represented by a linearly rising pulse front as a very good approximation (figure 2). This assumption is often accurate as the rise of the pulse is linear until before flattening out.

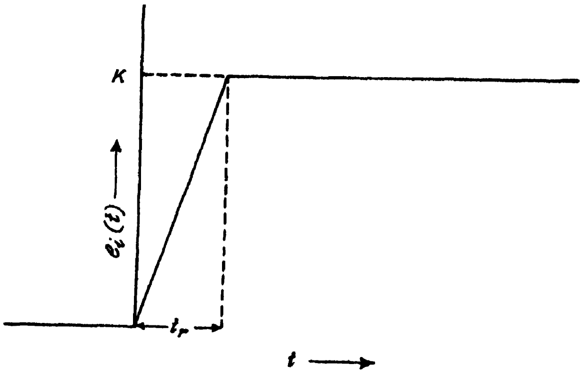

FIG. 2

A ramp function of height k with rise of time t .

The responses of cascaded differentiating and integrating circuits to a pulse (figure 2) which rises linearly upto a time t_r and then flattens out, will be obtained in the following analysis with the help of the theory of Laplace transforms. Such functions are called ramp functions.

It is shown that the shapes of differentiated and integrated pulses are much different in this case from those obtained when the input pulse is a step function. When the rise time is very short, the output pulse in both cases of differentiation and integration is practically similar to the output pulse when the input is as shown in figure 1.

Analysis of Integrating Circuits :

The pulse which is shown in figure 2 can be analytically represented by the following expression :

$$e_1(t) = \frac{Ku(t)t}{t_r} - \frac{Ku(t-t_r)(t-t_r)}{t_r} \qquad (2)$$

where, t_r indicates the rise time of the pulse and $u(t)$ and $u(t-t_r)$ are unit step functions (figure 1) beginning at times $t = 0$ and $t = t_r$ respectively. K represents the height of the pulse.

So the excitation transform which is the Laplace transform of $e_1(t)$ is given by

$$e_1(p) = \frac{K}{t_r}\left[\frac{1}{p^2} - \frac{\epsilon^{-pt_r}}{p^2}\right] \qquad (3)$$

The transfer function or system transform of n stages of cascaded integrating circuits (figure 3) is given by the equation (Bhattacharyya, 1952) :

Transfer function

$$\frac{e_n(p)}{e_1(p)} = \frac{1}{1+a_1Tp+\ldots\ldots+a_{n-1}T^{n-1}p^{n-1}+a_nT^np^n} \qquad (4)$$

where $e_n(p)$ and $e_i(p)$ are respectively the response transform and excitation transform

and

$$a_m = {}^{n+m}C_{n-m} = \frac{(n+m)!}{(n-m)!\ (2m)!} \tag{5}$$
and $T = RC$

FIG. 3
The n-stage RC-integrating circuit.

To avoid complicated results in obtaining the response $e_n(t)$ let us denote.

$$f_n(p) = \frac{K}{t_r} \cdot \frac{1}{p^2 (1 + a_1 Tp + a_2 T^2 p^2 + \ldots\ldots\ldots + T^n p^n)} \quad \ldots \quad (6)$$

so that the response $e_n(t)$ is given by

$$e_n(t) = f_n(t) - f_n(t - t_r)\ u(t - t_r) \tag{7}$$

So the problem is now to find out the inverse Laplace transform of $f\ (p)$. The expression of the nth degree in the denominator of equation (6) is to be broken up into linear factors so that the inverse Laplace transform can be found by applying Heaviside's expansion theorem. $f\ (p)$ can then be written as

$$f_n(p) = \frac{K}{t_r} \cdot \frac{1}{p^2 (Tp + a_1)(Tp + a_2)\ldots(Tp + a_n)} \quad \ldots \quad (8)$$

where $-a_i$'s are the roots of the polynomial

$$1 + a_1 Tp + \ldots\ldots + a_r T^r p^r + \ldots\ldots + T^n p^n \quad \ldots \quad (9)$$

It can be shown with the help of the theory of equations that the roots are distinct, irrational negative real numbers. Kenyon (1951) has given a table of the roots of the polynomial for values of $n = 1, 2, 3, 4$.

From the theory of Laplace transforms we know that if $x(p)$ is the transform of $x(t)$ *and* θ is a real positive number, then $\frac{1}{\theta}\, x(p/\theta)$ is the transform of $x(\theta t)$

Put $\theta = 1/T$

Then $Tx(Tp)$ is the transform of $x(t/T)$. Equation (8) can be written as

$$f_n(p) = \frac{KT}{t_r} \cdot \frac{T}{(Tp)^2 (Tp + a_1)(Tp + a_2)\ldots(Tp + a_n)} \tag{10}$$

So we can normalize equation (10) by substituting $t = t/Rc$ and $f_n(t) = \frac{f_n(t)}{K}$ to obtain

$$f_n(p) = \frac{T}{t_r} \cdot \frac{1}{p^2(p+a_1)\ (p+a_2)\ldots\ldots\ldots(p+a_n)} \qquad (11)$$

So the response function $f_n(t)$ that equals $Lt_r^{-1} f_r(p)$ is given by

$$f_n(t) = L^{-1} f_n(p)$$

$$= \frac{1}{2\pi j} \cdot \frac{T}{t_r} \cdot \int_{C-j\infty}^{C+j\infty} \frac{\epsilon^{pt}}{p^2\ (p+a_1)\ (p+a_2)\ \ldots\ (p+a_n)} \cdot dp \ldots \qquad (12)$$

where L^{-1} is the inverse Laplacian operator. We, therefore, have,

$$f_n(t) = \frac{T}{t_r} \cdot \Sigma \text{ Residues of poles of}$$

$$\frac{\epsilon^{pt}}{p^2\ (p+a_1)\ (p+a_2)\ \ldots\ (p+a_n)} \qquad (13)$$

by the Cauchy Residue theorem.

$f_n(t)$ has a pole of second order at $p=0$ with residue

$$\frac{d}{dp} \cdot \frac{\epsilon^{pt}}{(p+a_1)\ (p+a_2)\ \ldots\ (p+a_n)}\bigg|_{p=0} \qquad (14)$$

and poles of order unity at points $p = a_1, a_2, \qquad a_n$
with residues

$$\sum_{r=1} \frac{\epsilon^{-a_r t}}{\phi'(-a_r)} \qquad (15)$$

where

$$\phi(p) = p^2(p+a_1)\ (p+a_2) \qquad \ldots \qquad (p+a_n) \qquad (16)$$

and $\phi'(p)$ denotes differentiation of $\phi(p)$ (p) with respect to p

So with the help of the expressions (14) and (15), we have

$$f_n(t) = \frac{T}{t_r} \cdot \sum_{r=1}^{n} \frac{\epsilon^{-a_r t}}{\phi'(-a_r)} + \frac{T}{t_r} \cdot \frac{d}{dp} \cdot \frac{\epsilon^{pt}}{(p+a_1)\ \ldots\ (p+a_n)}\bigg|_{p=O} \qquad (17)$$

Equation (17) is expanded below for $n = 1, 2, 3, 4$.

$$f_1(t) = \frac{t}{t_r}(\epsilon^{-t} + t - 1) \qquad (18)$$

$$f_2(t) = \frac{T}{t_r}\left[3{\cdot}06522\ \epsilon^{-0{\cdot}38197\,t} - 0{\cdot}08271.\ \epsilon^{-2{\cdot}61808\,t} + t - 3\right] \qquad (19)$$

$$f_3(t) = \frac{T}{t_r}\left[6{\cdot}16180\ \epsilon^{-0{\cdot}19806\,t} - 0{\cdot}18013\ \epsilon^{-1{\cdot}55496\,t} + 0{\cdot}01838\ \epsilon^{-3{\cdot}24698\,t}\right.$$
$$\left. + t - 6\right] \qquad (20)$$

$$ {}_4(t)=\frac{T}{t_r}\Big[10{\cdot}29129\,\epsilon^{-0\cdot12061\,t}-0{\cdot}33333\,\epsilon^{-t}+0{\cdot}05106\,\epsilon^{-2\cdot84730\,t} $$

$$ -0{\cdot}00783\epsilon^{-8\cdot53209\,t}+t-10\Big] \qquad \ldots \quad (21) $$

Now the responses of multimesh RC integrating circuits can be found out with the aid of equations (7) and (18-21). The responses for sections from one to four are shown in figures 4—7 for three values of t_r/T to show distinctly the three cases

(*i*) $t_r/T < 1$ $(t_r/T=0{\cdot}5)$

(*ii*) $t_r/T = 1$

(*iii*) $t_r/T > 1$ $(t_r/T=1{\cdot}5)$

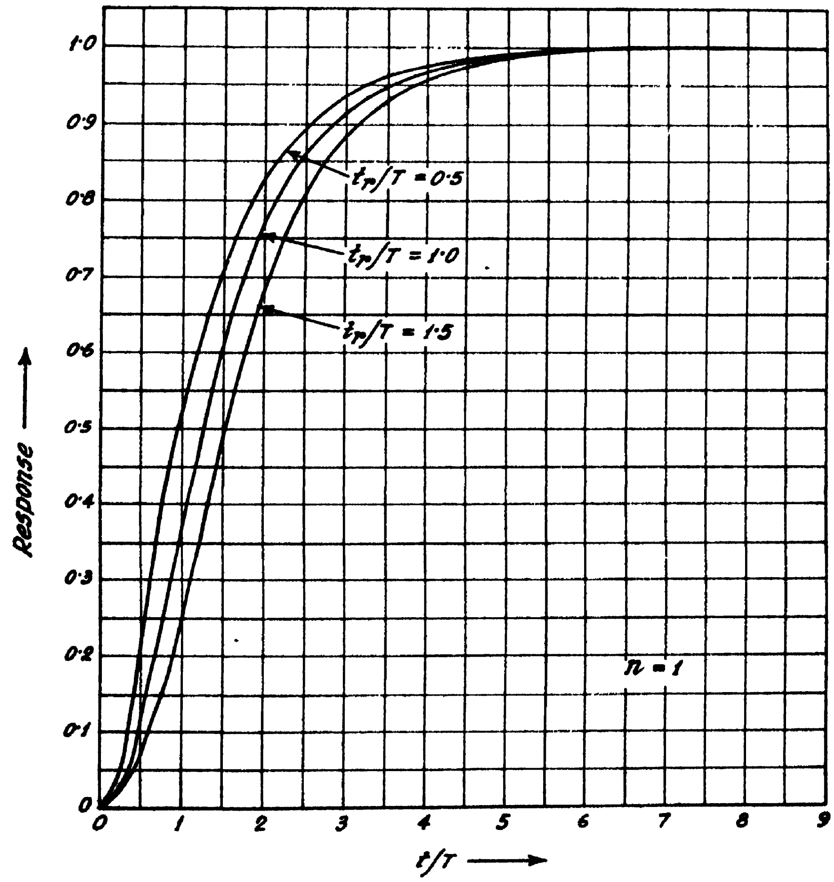

FIG. 4

Response to a ramp function of a single stage integrating circuit ($n=1$).

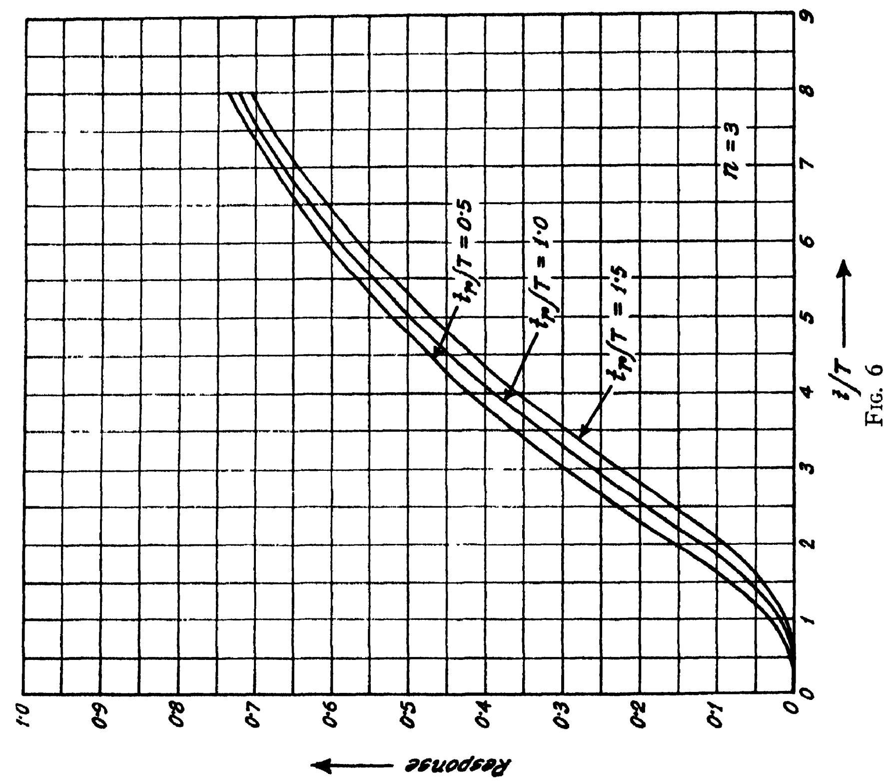

FIG. 6

Response to a ramp function of a three-stage integrating circuit ($n=3$).

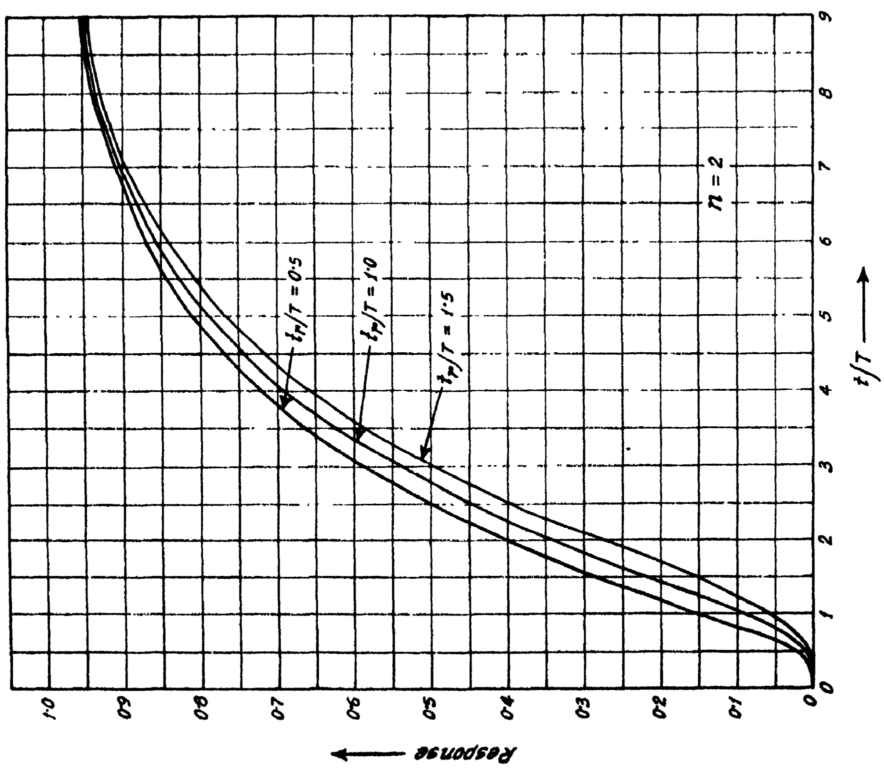

FIG. 5

Response to a ramp function of a two-stage integrating circuit ($n=2$).

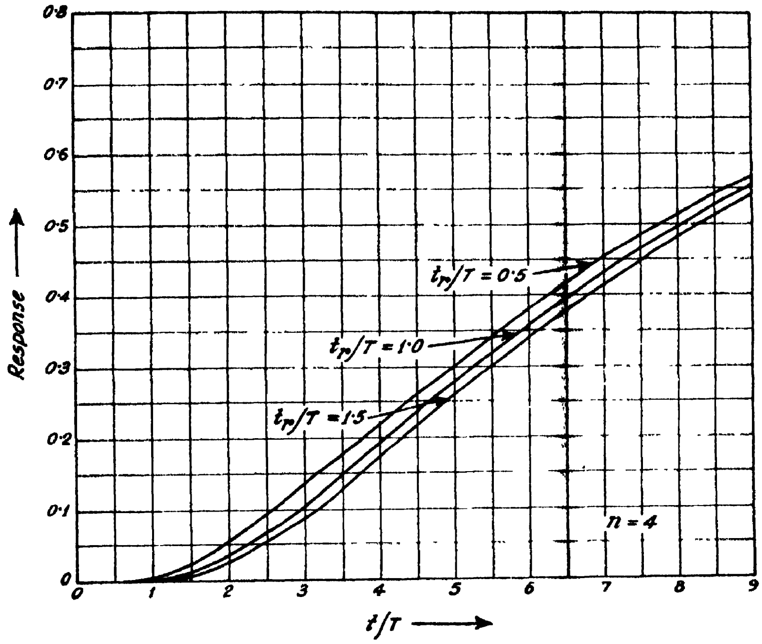

FIG. 7

Response to a ramp function of a four-stage integrating circuit ($n=4$).

It is seen from the figures that the output is dependent on the rise time t_r of the pulse. As t_r increases, the rate of increase of output is also diminished. With the increase of the number of sections the magnitude as well as the rate of rise of the output pulse sharply begin to fall.

We shall now study the case when the rise time is very short. It is natural that as the rise time is diminished, the output pulse-shape will be almost similar to that found in the case of step-function input pulse.

The excitation transform of the unit step-function pulse is given by

$$e_i(p) = \frac{1}{p} \qquad \ldots \quad (22)$$

So the transform of the output pulse is found to be

$$e_n(p) = \frac{1}{p} \cdot \frac{1}{1 + a_1 Tp + a_2 T^2 p^2 + \ldots + T^n p^n} \qquad \ldots \quad (23)$$

which can be expressed as

$$e_n(p) = \frac{1}{p} \cdot \frac{1}{(Tp + a_1)(Tp + a_2)\ldots(Tp + a_n)} \qquad \ldots \quad (24)$$

Normalizing equation (24) by putting $t = t/RC$ we obtain,

$$e_n(p) = \frac{1}{p} \cdot \frac{1}{(p + a_1)(p + a_2)\ldots(p + a_n)} \qquad (25)$$

Applying Heaviside's Expansion Theorem to equation (25), we obtain (Kenyon, 1951)

$$e_n(t) = \sum_{r=1}^{n} \frac{\epsilon^{-a_r t}}{f'(-a_r)} + \frac{1}{f'(0)} \qquad (26)$$

where

$$f(a_r) = p.\,(p + a_1)\ldots(p + a_r)\ldots(p + a_n) \qquad \ldots \quad (27)$$

The responses of cascaded integrating circuit to a step-function input voltage is now drawn in figure 8 with the help of equation (26).

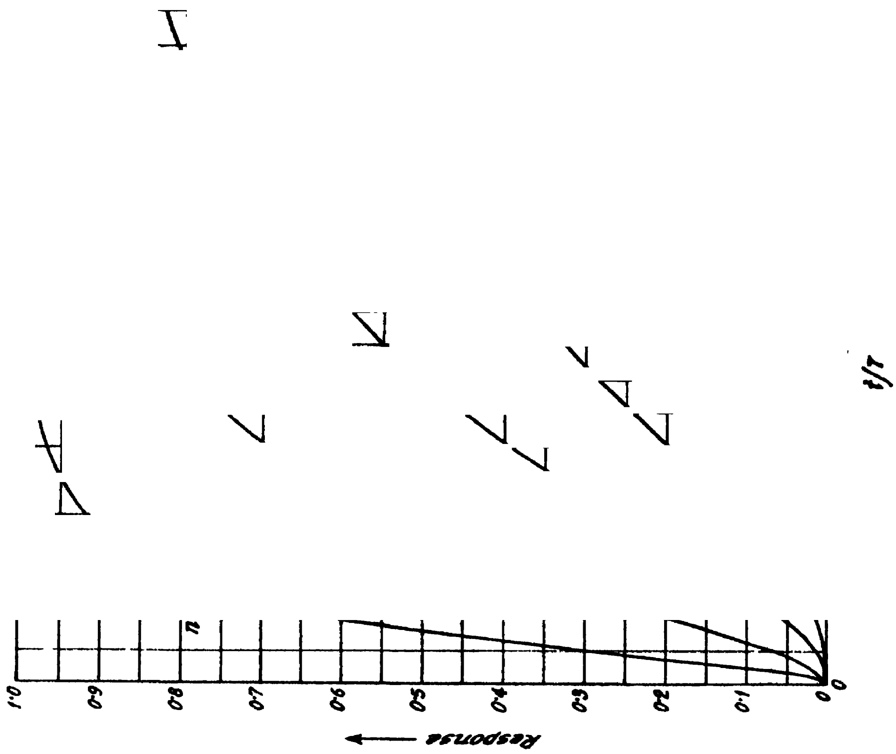

FIG. 8
Response to a step function input voltage of cascaded integrating circuits

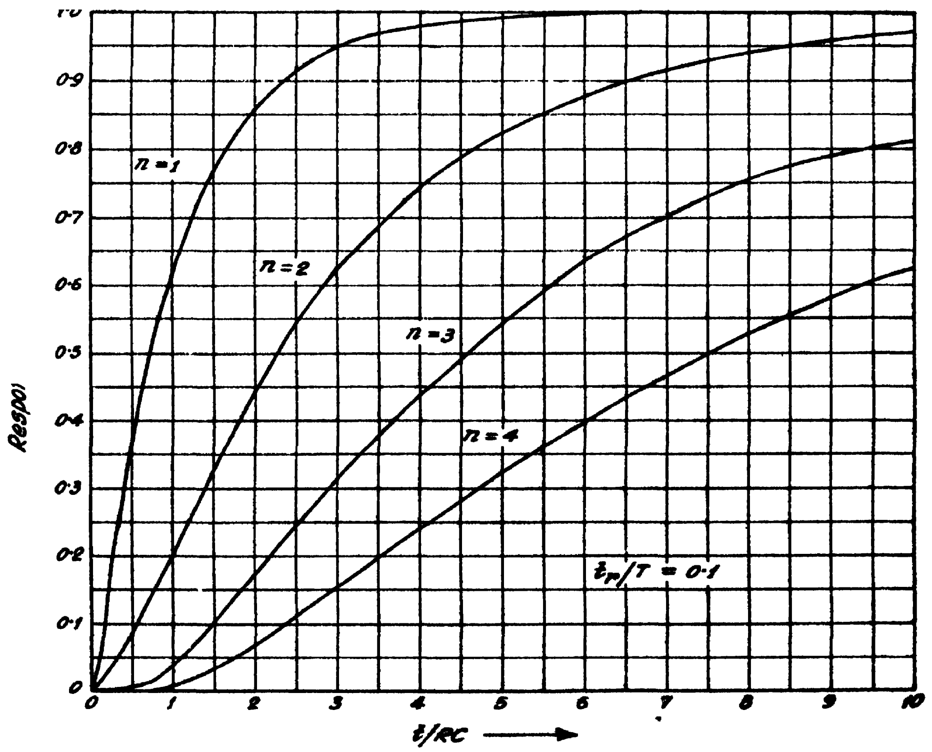

FIG. 9
Response to a ramp function of cascaded integrating circuits.

When the rise time is very short, e.g. $t_r/T = 0{\cdot}1$ the output pulses for various stages of integration are shown in figure 9.

It is evident from figures 8 and 9 that the response curves of integrating circuits to a pulse with a very short rise time is very much similar to the curves obtained in the case of step-function input voltage. In this case the assumption of a perfectly square pulse front is, no doubt, a good approximation of the sharply rising pulse.

Analysis of Differentiating Circuits :

The transfer function of n stages of cascaded differentiating circuits (figure 10) is given by the equation (Bhattacharyya, 1952) :

Transfer function:

$$\frac{e_n(p)}{e_i(p)} = \frac{(Tp)^n}{1+t_1Tp+\ldots+t_rT^rp^r+\ldots+T^np^n} \tag{28}$$

where,

$$t_r = {}^{2n-r}C_r = \frac{(2n-r)!}{r!\,(2n-2r)!} \tag{29}$$

and $T = RC$

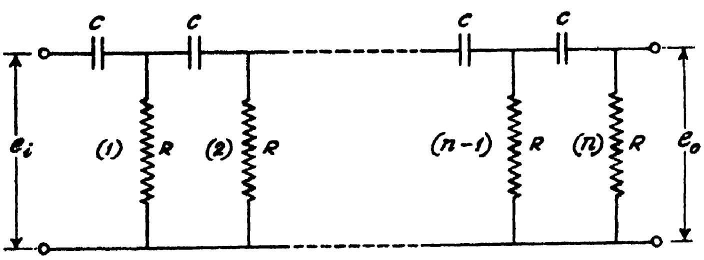

FIG. 10
A cascaded differentiating circuit.

If we denote

$$g_n(p) = \frac{K}{t_r}\cdot\frac{(Tp)^n}{p^2(1+t_1Tp+\ldots+t_rT^rp^r+\ldots+T^np^n)} \tag{30}$$

then the response $e_n(t)$ to a pulse given by equation (2) is given by

$$e_n(t) = g_n(t) - g_n(t-t_r).\ u(t-t_r) \quad \ldots (31)$$

It can be proved by the theory of equations that the roots of the polynomial

$$1+t_1Tp+\ldots\ldots+t_rT^rp^r+\ldots+T^np^n \quad \ldots (32$$

are the reciprocals of the roots of (9). So if $\beta_1, \beta_2, \ldots \beta_n$ denote the reciprocals of $\alpha_1, \alpha_2, , \ldots \alpha_n$, equation (30) can be written as

$$g_n(p) = \frac{K}{t_r}\cdot\frac{(Tp)^n}{p^2(Tp+\beta_1)(Tp+\beta_2)\ldots(Tp+\beta_n)} \quad \ldots (33)$$

As in the case of the integrating circuits, we can normalize (33) by putting $t = t/RC$ and $g_n(t) = g_n(t)/K$ to obtain

$$g_n(p) = \frac{T}{t_r} \cdot \frac{p^{n-2}}{(p+\beta_1)\,(p+\beta_2)\,\ldots\,(p+\beta_n)} \qquad \ldots \quad (34)$$

As the denominator of (34) is of higher degree in p than the numerator, Heaviside's expansion theorem can be applied to obtain the inverse transform

$$g_n(t) = \frac{T}{t_r} \sum_{r=1}^{n} \frac{\epsilon^{-\beta_r t}\,(-\beta_r)^{n-2}}{D'(-\beta_r)} \qquad (35)$$

where

$$D(p) = (p+\beta_1)\,(p+\beta_2)\,\ldots\,(p+\beta_r)\,\ldots\,(p+\beta_n) \qquad (36)$$

Equation (35) is expanded below for n from one to four.

$$g_1(t) = \frac{T}{t_r}\left[\,1-\epsilon^{-t}\,\right] \qquad (37)$$

$$g_2(t) = \frac{T}{t_r}\;0{\cdot}44721\left[\,\epsilon^{-0{\cdot}38197\,t} \quad \epsilon^{-2{\cdot}61803\,t}\,\right] \qquad (38)$$

$$g_3(t) = \frac{T}{t_r}\left[0{\cdot}43554\,\epsilon^{\;0{\cdot}64310\,t} - 0{\cdot}24171\,\epsilon^{-5{\cdot}04892\,t} - 0{\cdot}19384\,\epsilon^{-0{\cdot}30798\,t}\right] \ldots \quad (39)$$

$$g_4(t) = \frac{T}{t_r}\left[\,0{\cdot}09771\,\epsilon^{-0{\cdot}28312\,t} - 0{\cdot}28133\,\epsilon^{-0{\cdot}42602\,t} + 0{\cdot}33333\right.$$
$$\left. -\,0{\cdot}14969\,\epsilon^{\;\ldots 91\,t}\,\right] \qquad (40)$$

With the help of equations (31) and (37)— (40) the responses of cascaded differentiating circuits to a pulse as shown in figure 2 can be easily obtained The responses for sections from one to three are shown in figures 11—13 for three values of t_r/T as in the case of the analysis of integrating circuits.

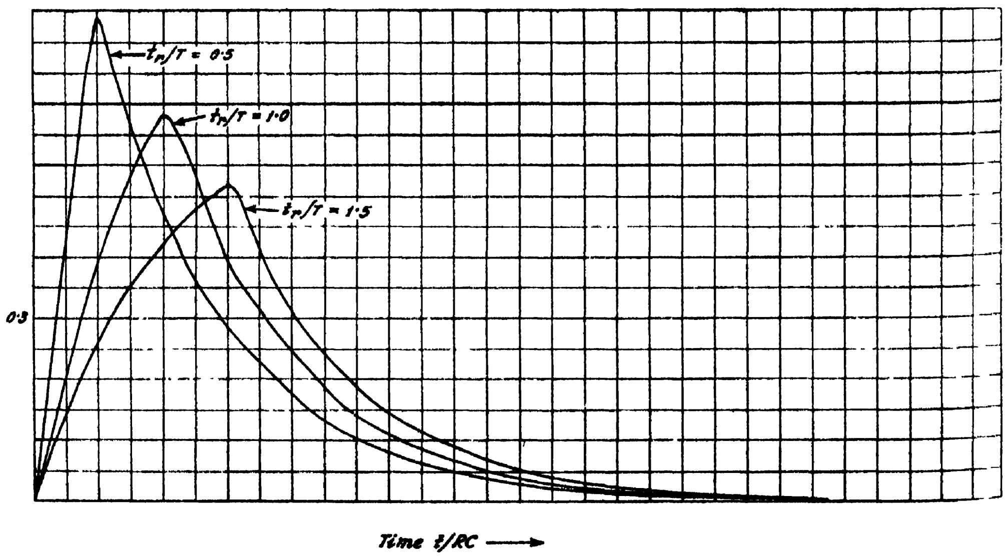

FIG. 11

Response to a ramp function of a single-stage differentiating circuit ($n=1$).

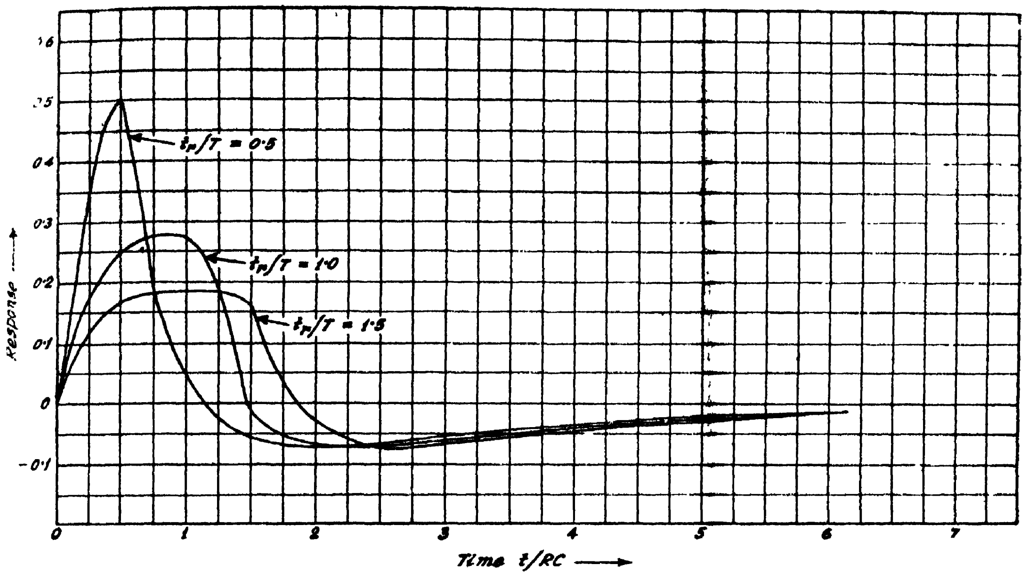

FIG. 12

Response to a ramp function of a two stage differentiating circuit ($n=2$).

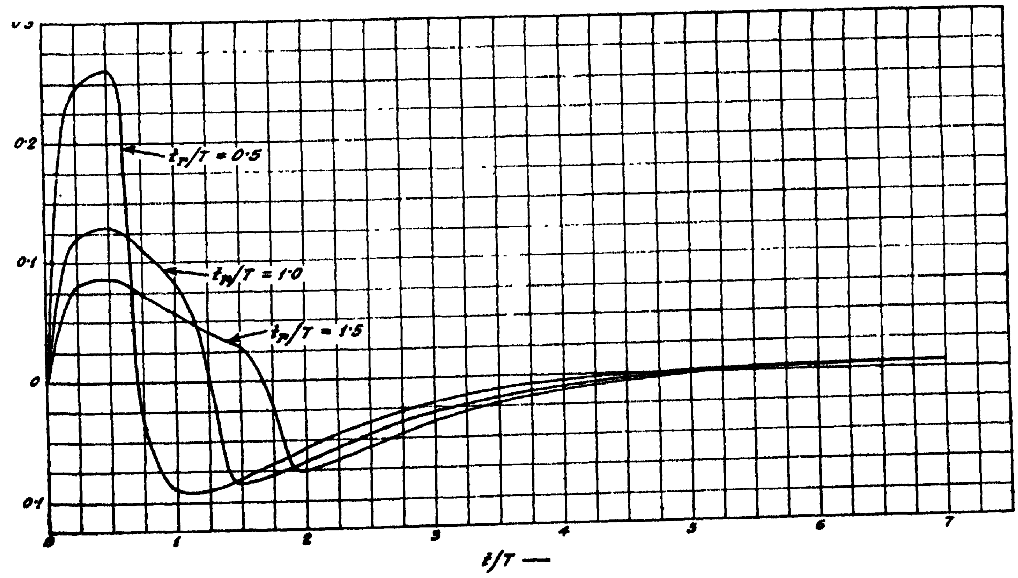

FIG. 13

Response to a ramp function of a three-stage differentiating circuit ($n=3$).

It is evident from the figures 11—14 that the characteristics of the output pulse shape depends very much upon two parameters: (i) the rise time t_r of the input pulse and (ii) the number of cascaded sections.

The effect of the rise time t_r can easily be seen from the following table which shows the maximum output voltage that can be obtained for an input pulse of specified rise time and the half amplitude widths of the output pulses in terms of t/RC.

$t_r/T=0$ (Step function input pulse):

n	1	2	3	4
Maximum output voltage ..	1	1	1	1
Half amplitude widths ..	0·690	0·225	0·100	0·051

$t_r/T=0.1$

n	1	2	3	4
Maximum output voltage ..	0·952	0·864	0·748	0·618
Half amplitude widths ..	0·75	0·26	0·16	0·13

$t_r/T=0·5$

n	1	2	3	4
Maximum output voltage ..	0·787	0·497	0·260	0·158
Half amplitude widths ..	0·95	0·53	0·25	0·21

$t_r/T=1·5$

n	1	2	3	4
Maximum output voltage ..	0·632	0·275	0·130	0·079
Half amplitude widths ..	1·32	1·05	0·51	0·31

$t_r/T = 1{\cdot}5$

	1			
Maximum output voltage	0·518	0·182	0·087	0·053
Half amplitude widths	1·68	1·55	0·56	0·28

From this table we find that the maximum amplitude of the output voltage varies directly with the rise time t_r. In the case of single section differentiating circuit, the maximum output voltage is obtained at $t = t_r$. But as the number of sections increases, the maximum output voltage occurs at an earlier time because of the rounded shape at the peak of the differentiated output in the case of $n = 1$. This is the reason why with the increase of n, the time at which the voltage response reaches its maximum becomes shorter.

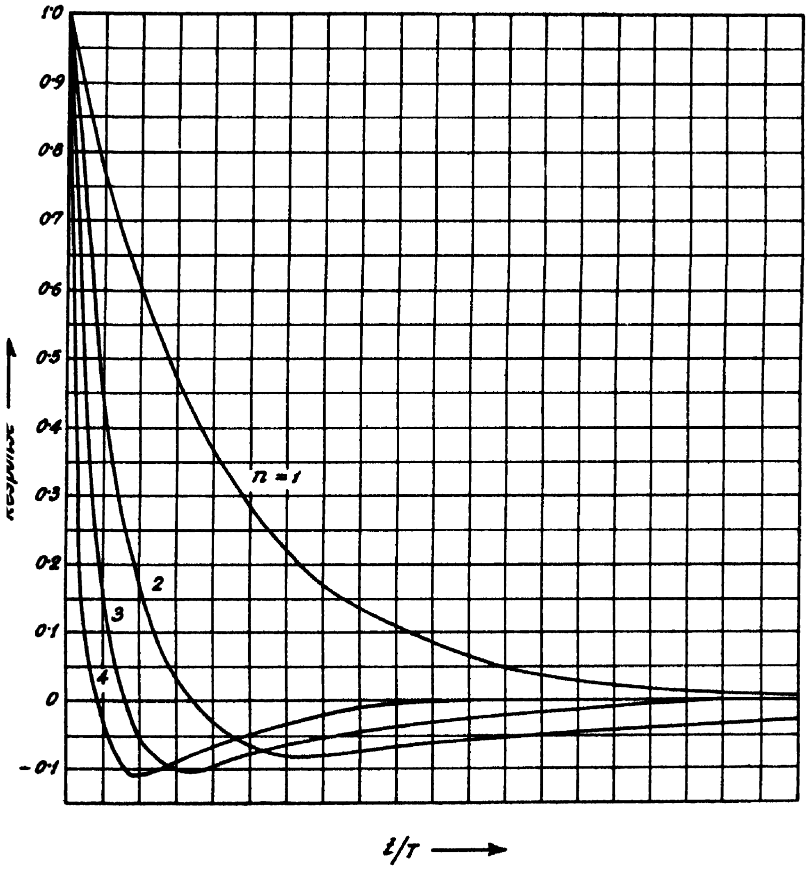

FIG. 14
Response to a step pulse of n-stage differentiating circuits.

A negative overshoot arises when $n > 1$ and it begins to oscillate from negative to positive value as the number of sections increases.

In the previous table we have also inserted the case when a unit step function voltage is applied to the input. The response can be found easily since we know both the system transform and excitation transform (Kenyon, 1951). The transform of the output voltage is then given by

$$e_n(p) = \frac{1}{p}\,\frac{(Tp)^n}{[1 + t_1 Tp + \ldots + t_r T^r p^r + \ldots + t_n T^n p^n]} \tag{41}$$

which can be written as

$$e_n(p) = \frac{1}{p}\,\frac{(Tp)^n}{(Tp+\beta_1)(Tp+\beta_2)\ldots(Tp+\beta_n)} \tag{42}$$

This can be normalized by substituting $t = t/RC$ to obtain

$$e_n(p) = \frac{p^{n-1}}{(p+\beta_1)(p+\beta_2)\ldots(p+\beta_n)} \tag{43}$$

By applying Heaviside's Expansion theorem we have,

$$e_n(t) = \sum_{r=1}^{n} \frac{(-\beta_r)^{n-1} e^{-\beta_r t}}{D'(-\beta_r)} \tag{44}$$

where

$D(p)$ is given by the expression (36).

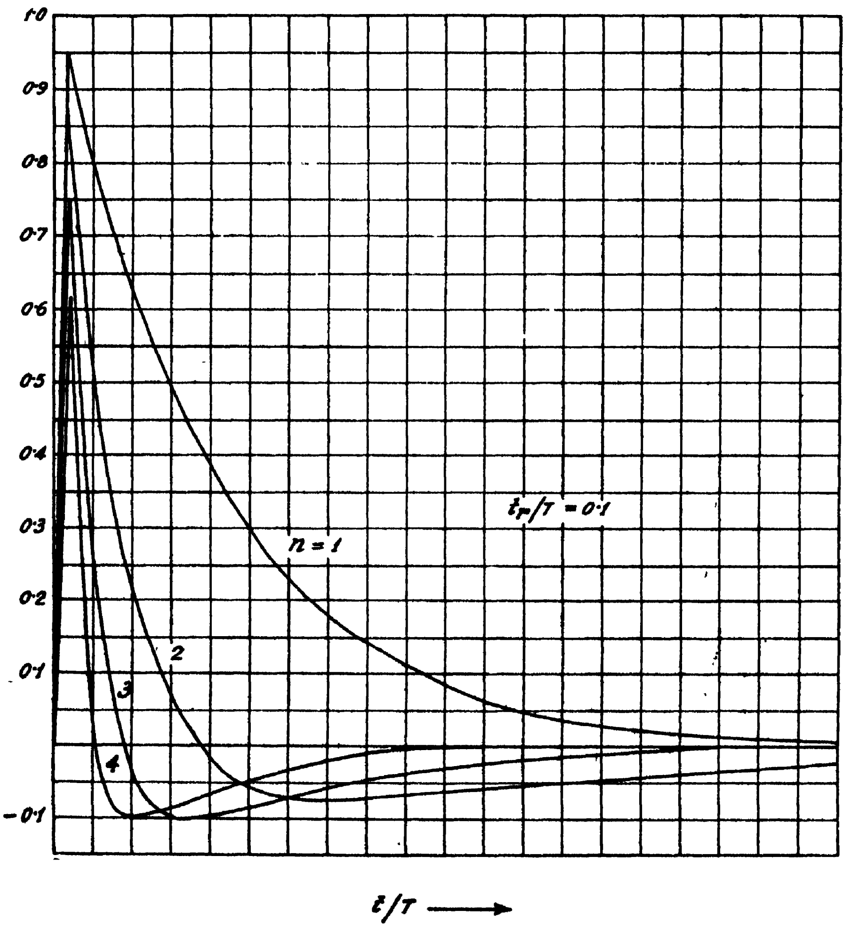

FIG. 15
Response to a ramp function of n-stage differentiating circuits.

So with the help of equation (44), we can find out the responses of cascaded differentiated circuits to step function input voltages as shown in figure 15.

As the rise time t_r will be diminished, the responses will gradually be identical with that in figure 15. This can clearly be seen if we compare figures 14 and 15 where figure 15 is drawn for the case $t_r/T = 0.1$ Though in the latter case the maximum output voltage drops very much when $n > 1$,, the exponential fall of the two curves shows marked similarity.

It is seen from figure 15 that even when $n > 1$, the maximum output voltage occurs at time $t = t_r$ because of the short rise time and the absence of rounded peak.

CONCLUSION

The present analysis of integrating and differentiating circuits has been done in order to show the importance of rise time of the input pulse in modifying their responses. In the case of integration the major effect of the rise time is to delay the integrated output to reach its maximum value. While in differentiation, the rise time practically determines the maximum output voltage that can be obtained. This limitation may become serious when the differentiated response is applied for the purpose of triggering a circuit.

Again, the flat top of the output that has been shown in some cases, e.g. in figures and , is sometimes very unsuitable for triggering.

ACKNOWLEDGMENTS

The author is indebted to Prof. M. N. Saha, F.R.S. and Dr. B. D. Nag for their kind interest in the work. Sincere thanks are also due to Mr. B. M. Banerjee and Dr. A. K. Saha for valuable discussions.

REFERENCES

Bhattacharyya, B. K., 1952, *Ind. J. Phys.*, **26**, 563.
Kenyon, R. R., 1951, *Proc. I. R. E.*, **39**, 557.
Rubin, M. D., 1952, Electronics for Communication Engineers, McGraw-Hill Book Co.
Schwartz, L. S., 1946,—*Proc. I. R. E.*, **34**, 862.